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Spectral Properties of a Periodically Kicked Quantum Hamiltonian

M. Combescure[1]

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We study the spectral properties of the Floquet operator for the periodically kicked Hamiltonian $H(t)=H_0+\lambda\varphi\rangle\langle\varphi\sum_{+\infty}^{-\infty}\delta(t-nT)$, H_0 being self-adjoint and pure point. We show that the Floquet operator is pure point for almost every λ, if φ is cyclic for H_0 and has absolutely convergent expansion in the basis of eigenstates of H_0. When this last condition is not satisfied, the Floquet operator can have a continuous spectrum, as we show by an example.

KEY WORDS: Quantum stability problem; periodically kicked systems.

1. INTRODUCTION

There is a growing interest in the study of quantum systems with time-dependent Hamiltonians. An important motivation in this area is a better understanding of the quantum dynamics of simple systems whose classical counterpart exhibits chaotic behavior. Among these simple systems are the time-periodic ones, like the "kicked rotator,"[(4)] the hydrogen atom in electromagnetic fields,[(1)] and the trapping of charged particles in quadrupolar radiofrequency traps.[(7)] It is known that all these systems undergo, classically, a transition from regular to chaotic motion as the strength of the time-periodic perturbation exceeds some critical value. On the other hand, one often encounters time-periodic (or quasiperiodic) systems that are purely quantum systems, like, say, two-state systems in quantum optics; their long-time behavior has recently been the subject of an intense investigation.[(3,5,8,10,16–19)] All these approaches have in common the following set of questions:

[1] Laboratoire de Physique Théorique et Hautes Energies (Laboratoire associé au Centre National de la Recherche Scientifique), Université de Paris XI, 91405 Orsay Cedex, France.

1. Are there two qualitatively different dynamical quantum behaviors under these time-dependent perturbations?

2. Is it possible to characterize quantitatively these dynamical behaviors as "regular" or "irregular"?

3. Is there, as in classically chaotic systems, a "loss of memory" of the initial state under the evolution?

4. Which parameters of the system govern a possible change in the dynamics (from a "regular" to an "irregular" one)?

5. Are there "resonance phenomena" for suitable time frequencies?

6. What happens when the strength of the time-dependent perturbation increases?

For the hydrogen atoms as well as the "kicked rotator," question 1 has been approached through numerical experiments, allowing a detailed comparison between the classical and quantum dynamics. Roughly speaking, these systems exhibit a quantum suppression of the classical diffusive behavior associated with the classically chaotic regime.[3-5] On the contrary, a similar treatment of quasiperiodically driven quantum systems exhibits a more "irregular" dynamical behavior when the two driving frequencies are incommensurate.[16,17] However, in both cases the system seems dynamically stable in the sense that it suitably "keeps the memory" of the initial state in practical computations (this means that the infinitesimal errors inherent to numerical computations do not add up in the irregular regime to destroy irreversibility).[5] Although the "large times" at which the numerical experiments have been performed might not be large enough to show a possibly later loss of memory of the initial state, these results seem to be a strong indication that the answer to question 3 is "no" in these cases.

In the time-periodic case as well as in the quasiperiodic case with commensurate frequencies, the Floquet analysis provides one with a convenient self-adjoint operator called the "quasi-energy operator"; it can be viewed as a photon–matter Hamiltonian with a photon–matter coupling originating from the time-dependent term.[21] The spectral properties of this quasi-energy operator appear to be a suitable tool for studying the long-time behavior of such systems, allowing one to give to question 2 a precise answer: eigenstates of the quasi-energy operator correspond to a regular long-time behavior which is recurrent, whereas states of the continuous spectral subspace exhibit some kind of diffusive behavior in phase space (which is close in spirit to the well-known RAGE theorem).[21,9]

With this characterization in mind, several authors have investigated a possible transition in the nature of the quasi-energy spectrum of the kicked-rotator model when suitable parameters of the system are varied: going to dimensionless variables, the two relevant parameters in the quantum problem appear to be

$$\alpha = \hbar T/4\pi \qquad (T \text{ is the time period})$$

and

$$\lambda = kT \qquad (k \text{ is the coupling constant})$$

The resonant case $\alpha \in \mathbb{Q}$ yields an absolutely continuous spectrum (no matter how large the dimensionless coupling λ is),[14] and the quasi-energy spectrum still is continuous when α is close enough to rationals (Liouville number).[6] This answers question 5 for this problem. In the remaining (generic) case, no rigorous answer has been given. However, Bellissard[2] has proven for a smoothed version of the kicked rotator that the quasi-energy spectrum is pure point for small enough coupling λ and for nonresonant values of α. Similarly, for time-periodic perturbations of the harmonic oscillator (a question inherited from the stability problem in quadrupolar radiofrequency traps), I have proven that the quasi-energy spectrum is pure point provided the dimensionless coupling is small enough and the dimensionless frequency suitably nonresonant.[7] This strongly suggests that α and λ are two parameters that govern a possible change in the dynamics (question 4). However, for these simple time-periodic systems, we have not been able to go beyond the small coupling regime, which leaves question 6 open in these cases.

Nevertheless, there are recent results by Howland[11-13] that shed a new light on the quantum stability problem under time-periodic perturbations. Howland presents a large class of time-periodic perturbations $V(t)$ such that, if H is any discrete, positive Hamiltonian for which the gap between two consecutive eigenvalues λ_k and λ_{k-1} increases like k^α ($\alpha > 0$), then the quasi-energy spectrum for $H + \lambda V(t)$ is pure point for almost all values of the coupling λ (no matter how large!). In this result, only the parameter λ is relevant for question 4, but the result is *nonperturbative* (question 6), and there is *no resonance phenomenon* (question 5), in contrast to the previous cases. Not surprisingly, the kicked rotator (too singular in t!) and the harmonic oscillator ($\alpha = 0$) are not covered by this approach!

In this paper, we concentrate on the quantum dynamics of the following periodically kicked Hamiltonian:

$$H(t) = H_0 + \lambda\varphi\rangle\langle\varphi \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

H_0 is an arbitrary self-adjoint operator in a Hilbert space $\mathscr{H}$, with pure point spectrum, λ is a real coupling, and φ is a state in $\mathscr{H}$. We prove the following result [let V_λ be the Floquet operator for $H(t)$, i.e., $V_\lambda = U(T, 0)$, $U(t, s)$ being the unitary evolution operator generated by $H(t)$].

Theorem. Let H_0 be pure point with eigenstates $(\varphi_n)_{n \in \mathbb{Z}}$ and let

$$\varphi = \sum_{-\infty}^{+\infty} a_n \varphi_n \qquad \left(\sum_{-\infty}^{+\infty} |a_n|^2 = 1 \right)$$

(i) Assume φ is cyclic for H_0, and

$$\sum_{-\infty}^{+\infty} |a_n| < \infty$$

Then V_λ has pure-point spectrum for almost every λ.

(ii) Let H_0 be the Hamiltonian of the harmonic oscillator with frequency ω, and assume $a_n = 2\pi n^{-\gamma}$ with $1/2 < \gamma < 1$. Then if $\omega T/2\pi$ is diophantine, φ belongs to the continuous spectral subspace of V_λ (any λ).

We want to add some comments before giving the proof of this simple result in the following section.

Remark a. Even very simple rank-one perturbations can produce a continuous quasi-energy spectrum.

Remark b. Result (i) is a nonperturbative quantum stability result which holds whether the time frequency is resonant or not. On the contrary, the instability result of (ii) (again nonperturbative) is a true *nonresonant* phenomenon, because it holds when $\omega T/2\pi$ is irrational (in the resonant case $\omega T/2\pi \in \mathbb{Q}$, the Floquet operator has pure-point spectrum, in contrast to the usual "kicked rotator"; this follows from the invariance of the essential spectrum in this case).

Remark c. We conjecture that (ii) can be generalized to unperturbed Hamiltonians H_0 with eigenvalues λ_n of the form

$$\lambda_n = \hbar \sum_{0}^{p} \alpha_j n^j$$

provided $\alpha_j T/2\pi$ is diophantine for some j: $1 \leqslant j \leqslant p$.

2. PROOF OF THE THEOREM

The unitary operator that connects the state "just before the nth kick" to the state "just before the $(n-1)$th kick" is[14]

$$e^{+iTH_0/\hbar} e^{+i\lambda T \varphi \rangle \langle \varphi / \hbar} \tag{1}$$

It is independent of n, and, in order to avoid questions of definiteness with the "δ functions," we take (1) as a definition of the Floquet operator V_λ. We denote

$$U = e^{+iTH_0/\hbar} \tag{2}$$

$$\mu = e^{+i\lambda T/\hbar} - 1 \tag{3}$$

and an elementary computation yields

$$V_\lambda = U(1 + \mu\varphi\rangle\langle\varphi) \tag{4}$$

Therefore we are in a situation very similar to that considered in ref. 20, with unitary instead of self-adjoint operators. The spectral theorem for unitary operators yields[22]

$$V_\lambda = \int_0^{2\pi} e^{-i\theta}\, dF_\lambda(\theta) \tag{5}$$

so that, denoting

$$d\mu_\lambda(\theta) = \langle\varphi, dF_\lambda(\theta)\,\varphi\rangle \tag{6}$$

we have, for any z such that $|z| \neq 1$,

$$F_\lambda \equiv \langle\varphi, (V - z)^{-1}\,\varphi\rangle = \int_0^{2\pi} \frac{d\mu_\lambda(\theta)}{e^{i\theta} - z} \tag{7}$$

Define

$$B(x) = \left[\int_0^{2\pi} d\mu_0(\theta)\left(\sin^2\frac{x-\theta}{2}\right)^{-1}\right]^{-1} \tag{8}$$

Then we have the following result.

Proposition 1. Let $\lambda \neq 0$. Then $d\mu_\lambda$ has a pure point at $x \in [0, 2\pi)$ if an only if

$$\lim_{\varepsilon \searrow 0} e^{i(x+i\varepsilon)} F_0(e^{i(x+i\varepsilon)}) = -\frac{1+\mu}{\mu}, \qquad B(x) \neq 0$$

Proof. By the first resolvent identity, we have, for any $z \in \mathbb{C}$ such that $|z| \neq 1$,

$$(U - z)^{-1} = (V - z)^{-1} + \mu\varphi\rangle\langle\varphi(V - z)^{-1} + z\mu(U - z)^{-1}\,\varphi\rangle\langle\varphi(V - z)^{-1}$$

so that, taking expectation values with φ,

$$F_\lambda(z)=F_0(z)[1+\mu+z\mu F_0(z)]^{-1} \tag{9}$$

But it is clear that if e^{ix} is an eigenvalue of V_λ, then

$$F_\lambda(e^{i(x\pm i\varepsilon)}) \underset{\varepsilon\searrow 0}{\sim} \pm\varepsilon^{-1}e^{-ix}\mu_\lambda(\{x\})\to\pm\infty \tag{10}$$

and therefore, due to (9),

$$\lim_{\varepsilon\searrow 0} e^{i(x\pm i\varepsilon)}F_0(e^{i(x\pm i\varepsilon)})=-\frac{1+\mu}{\mu} \tag{11}$$

Moreover, denoting $F_\lambda^\pm=F_\lambda(e^{i(x\pm i\varepsilon)})$, we have

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\varepsilon^{-1}\left(\frac{F_0^+}{F_\lambda^+}-\frac{F_0^-}{F_\lambda^-}\right)&=\mu\lim_{\varepsilon\searrow 0}\varepsilon^{-1}(e^{i(x+i\varepsilon)}F_0^+-e^{i(x-i\varepsilon)}F_0^-)\\&=\mu\lim_{\varepsilon\searrow 0}\int_0^{2\pi}\frac{d\mu_0(\theta)\,e^{i(x+\theta)}(e^{-\varepsilon}-e^{\varepsilon})}{(e^{i\theta}-e^{ix})^2-\varepsilon^2e^{i(x+\theta)}}\\&=\mu[2B(x)]^{-1}\end{aligned} \tag{12}$$

But the lhs of (12) equals

$$\lim_{\varepsilon\searrow 0}\frac{\varepsilon^{-1}(F_0^+-F_0^-)}{F_\lambda^+}+\lim_{\varepsilon\searrow 0}F_0^-[(\varepsilon F_\lambda^+)^{-1}-(\varepsilon F_\lambda^-)^{-1}] \tag{13}$$

If $B(x)\neq 0$, the first term of (13) is obviously zero, whereas the second one is

$$-e^{-ix}\frac{1+\mu}{\mu}2[e^{-ix}\mu_\lambda(\{x\})]^{-1}=\frac{-2(1+\mu)}{\mu\mu_\lambda(\{x\})}$$

Therefore

$$\mu_\lambda(\{x\})=\frac{-4(1+\mu)}{\mu^2}B(x) \tag{14}$$

and the converse statement is easily obtained.

Corollary 2. Assume H_0 is pure point, with $\{\varphi_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ as eigenstates and eigenvalues, respectively, and let $\varphi=\sum_0^{+\infty}a_n\varphi_n$ be a cyclic vector for H_0 with norm unity. Then e^{ix} belongs to the point spectrum of V_λ if and only if

$$B(x)^{-1} = \sum_0^{+\infty} \frac{|a_n|^2}{\sin^2[(x-\theta_n)/2]} < \infty \tag{15}$$

and

$$\sum_0^{+\infty} |a_n|^2 \operatorname{cotg} \frac{x-\theta_n}{2} = \operatorname{cotg} \frac{\lambda T}{2\hbar} \tag{16}$$

where

$$\theta_n = 2\pi\{\lambda_n T/2\pi\hbar\} \tag{17}$$

$\{x\}$ being the fractional part of x.

Proof. By the cyclicity of φ, e^{ix} is an eigenvalue of V_λ if and only if $d\mu_\lambda(\theta)$ has a pure point at $\theta = x$. But using

$$d\mu_0(\theta) = \sum_0^{+\infty} |a_n|^2 \, \delta(\theta - \theta_n)\, d\theta$$

it is clear that the second condition of Proposition 1 reduces to (15), and that (16) follows from the first condition by equating the real and imaginary parts (and using the normalization $\sum_0^{+\infty} |a_n|^2 = 1$). Note that (15) and the normalization condition imply that (16) is absolutely convergent.

In order to complete the proof of part (i) of the theorem, we need the following lemmas.

Lemma 3. Assume $\sum_0^{+\infty} |a_n| < \infty$. Then (15) is true for almost every $x \in \mathbb{R}$ (regardless of the sequence $\{\lambda_n\}$).

For the proof, see ref. 11, Section 3.

Lemma 4. The following statements are equivalent:

(a) For a.e. λ, V_λ has only point spectrum.

(b) For a.e. x, $B(x) \neq 0$.

Proof. It is easy to see, using Cayley transforms, that the continuous part of the measure $d\mu_\lambda$ is supported outside $E = \{x \in [0, 2\pi) \colon B(x) \neq 0\}$. (See ref. 20 for the corresponding statement for self-adjoint operators.) But we have just seen that, for any $\lambda \neq 0$, μ_λ^{pp} (the pure-point spectral measure for V_λ) is supported by E. Thus, V_λ has only pure-point spectrum for almost every λ if and only if $\mu_\lambda([0, 2\pi)\backslash E) = 0$ for a.e. λ. This in turn holds if and only if

$$\int_0^{2\pi} d\lambda' \, h(\lambda') \, \mu_\lambda([0, 2\pi)\backslash E) = 0 \tag{18}$$

λ' being the dimensionless constant

$$\lambda' = \lambda T/\hbar \tag{19}$$

and

$$h(\lambda) = 2 \operatorname{Re} \frac{1}{1 - ce^{i\lambda}} \tag{20}$$

for some $|c| < 1$.

But we show below (Lemma 5) that the measure η defined by

$$\eta(X) = \int_0^{2\pi} d\lambda' \, h(\lambda') \, \mu_\lambda(X) \tag{21}$$

{for X a Borel set in $[0, 2\pi)$} is equivalent to the Lebesgue measure. Thus, $\eta([0, 2\pi)\backslash E) = 0$ implies that a.e. $x \in [0, 2\pi)$ belongs to E, i.e., satisfies $B(x) \neq 0$.

Lemma 5. The measure η defined by (20)–(21) is equivalent to the Lebesgue measure.

Proof. The proof is analogous, for unitary operators, to an argument of Simon and Wolff[20] for self-adjoint operators. Define, for $z \in \mathbb{C}$ with $|z| \neq 1$,

$$H(z) = z \int_0^{2\pi} \frac{d\eta(\theta)}{e^{i\theta} - z} = \int_0^{2\pi} d\lambda' \, h(\lambda') = F_\lambda(z) \tag{22}$$

It is clear (as in the proof of Proposition 1) that $[H(e^{i(x + i\varepsilon)}) - H(e^{i(x - i\varepsilon)})] \, dx$ converges weakly to $d\eta$ as $\varepsilon \searrow 0$. But defining for a.e. $x \in [0, 2\pi)$

$$\operatorname{cotg} \frac{\alpha(x)}{2} = \sum_0^{+\infty} |a_n|^2 \operatorname{cotg} \frac{x - \lambda_n}{2} \tag{23}$$

it is easy to see that

$$\left. \frac{zF_0}{1 + zF_0} \right|_{Z = \exp[i(x \pm i\varepsilon)]} \underset{\varepsilon \searrow 0}{\sim} e^{i\alpha(x)} \left(1 \mp \varepsilon B(x)^{-1} \sin^2 \frac{\alpha(x)}{2} \right)$$

from which we get the following Fourier expansions or $zF_\lambda(z)$:

$$z=e^{i(x+i\varepsilon)}: \qquad zF_\lambda=\sum_1^\infty e^{-in\lambda'}\left(\frac{zF_0(z)}{1+zF_0(z)}\right)^n \tag{24}$$

$$z=e^{i(x-i\varepsilon)}: \qquad zF_\lambda=-\sum_0^\infty e^{in\lambda'}\left(\frac{1+zF_0(z)}{zF_0(z)}\right)^n \tag{25}$$

$$\left.\frac{1+zF_0}{zF_0}\right|_{z=\exp[i(x-i\varepsilon)]}=\left.\left(\frac{zF_0}{1+zF_0}\right)^*\right|_{z=\exp[i(x+i\varepsilon)]} \tag{26}$$

Then, by periodicity in λ and the Bessel–Parseval relation, we get, using (24),

$$\int_0^{2\pi} h(\lambda')\,d\lambda'\,zF_\lambda|_{z=\exp[i(x+i\varepsilon)]}=\left.\frac{czF_0/(1+zF_0)}{1-czF_0/(1+zF_0)}\right|_{z=\exp[i(x+i\varepsilon)]}$$

and similarly, due to (25),

$$\int_0^{2\pi} h(\lambda')\,d\lambda'\,zF_\lambda|_{z=\exp[i(x-i\varepsilon)]}=\left.\frac{-\bar{c}(1+zF_0)/zF_0}{1-\bar{c}(1+zF_0)/zF_0}\right|_{z=\exp[i(x-i\varepsilon)]}$$

Therefore, using (26),

$$\begin{aligned} H(e^{i(x+i\varepsilon)})-H(e^{i(x-i\varepsilon)}) &= 2\,\mathrm{Re}\left.\frac{czF_0/(1+zF_0)}{1-czF_0/(1+zF_0)}\right|_{z=\exp[i(x+i\varepsilon)]} \\ &\underset{\varepsilon\searrow 0}{\sim} \frac{\rho_\varepsilon(x)[\cos\alpha(x)-\rho_\varepsilon(x)]}{1-2\rho_\varepsilon(x)\cos\alpha(x)+\rho_\varepsilon^2(x)}\equiv h_\varepsilon(x) \end{aligned}$$

where $\rho_\varepsilon(x)=c\exp[-\varepsilon B(x)^{-1}\sin^2\alpha(x)/2]$.

Now it is easy to check that

$$-2\leqslant h_\varepsilon(x)\leqslant 1$$

uniformly for $x\in[0,2\pi)$ and $\varepsilon>0$ small enough. This shows that $d\eta$ is absolutely continuous with respect to the Lebesgue measure. But the converse also holds because

$$\{x\colon \lim_{\varepsilon\searrow 0} h_\varepsilon(x)=0\}=\{x\colon \cos\alpha(x)=c\}$$

has zero Lebesgue measure by standard results on analytic functions [because $\cos\alpha(x)$ can be viewed as the boundary value of an analytic function]. This completes the proof.

We now go to the proof of part (ii) of the theorem. Here we no longer need the cyclicity assumption because, if the measure $d\mu_\lambda$ is continuous, then the continuous spectral subspace for V_λ contains at least the state φ. By assumption we have (replacing H_0 by $H_0 + \hbar\omega/2$, which is harmless)

$$\lambda_n = n\hbar\omega \tag{27}$$

We have seen that $B(x) \neq 0$ is a necessary and sufficient condition for the spectral measure $d\mu_\lambda$ to be pure point at e^{ix}. But

$$B(x)^{-1} = \sum_0^\infty \frac{|a_n|^2}{\sin^2[(x-\theta_n)/2]} \geqslant \sum_0^\infty |a_n|^2 \left(\frac{2}{x-\theta_n}\right)^2$$
$$\geqslant \sum_{n \in S(x)} \frac{4\,|a_n|^2}{(x-\theta_n)^2} \geqslant 4\ \#S(x)$$

where $S(x)$ is by definition

$$S(x) = \{n: |x-\theta_n| \leqslant |a_n| = n^{-\gamma}2\pi\} \tag{28}$$

By assumption, $\omega T/2\pi \notin \mathbb{Q}$, so that the sequence $\theta_n/2\pi = \{n\omega T/2\pi\}$ is uniformly distributed modulo 1.[15] Moreover, the worse $\omega T/2\pi$ is approximated by rationals, the more uniform is the distribution of $\theta_n/2\pi$, and we have the following result.

Lemma 6. Let $\omega T/2\pi$ be diophantine. Then, for any $x \in [0, 2\pi)$, the number $\#S(x)$ of elements of $S(x)$ is infinite.

Proof. Given $(\mu_n)_{n \in \mathbb{N}}$, $\mu_n \in [0, 1)$, we define for $0 \leqslant \alpha < \beta \leqslant 1$,

$$A([\alpha, \beta), N) = \#\{n \leqslant N: \mu_n \in [\alpha, \beta)\}$$

and we introduce the so-called "discrepancy" D_N of the sequence μ_n:

$$D_N = \sup_{0 \leqslant \alpha < \beta \leqslant 1} \left| \frac{A([\alpha, \beta), N)}{N} - (\beta - \alpha) \right|$$

which measures the lack of uniformity in the distribution of $(\mu_n)_{n \leqslant N}$. For every $x \in (0, 2\pi)$, the interval

$$J_N(x) = \left[\frac{x}{2\pi} - N^{-\gamma}, \frac{x}{2\pi} + N^{-\gamma}\right)$$

is contained in $[0, 1]$ for N large enough and we have

$$|N^{-1}A(J_N(x), N) - 2N^{-\gamma}| \leqslant D_N$$

But if $\omega T/2\pi$ is diophantine, or at least is of constant type (see ref. 15), we have $ND_N = O(N^{\varepsilon})$ for every $\varepsilon > 0$, and therefore

$$A(J_N(x), N) \geqslant N^{1-\gamma}$$

for N large enough. Now taking $\mu_n = \theta_n/2\pi$, it is clear that

$$\{n \leqslant N: |x - \theta_n| \leqslant 2\pi n^{-\gamma}\} \supset \{n \leqslant N: \mu_n \in J_N(x)\}$$

and therefore, using (29),

$$\begin{aligned} \# S(x) &= \overline{\lim_N}\ \# \{n \leqslant N: |x - \theta_n| \leqslant 2\pi n^{-\gamma}\} \\ &\geqslant \overline{\lim_N}\ A(J_N(x), N) = \infty \end{aligned}$$

This completes the proof of Lemma 6, and thus of part (ii) of the theorem, because $d\mu_\lambda$ is continuous for every $\lambda > 0$.

The continuous spectrum of V_λ may, *a priori*, be absolutely continuous, in contrast to the case of rank-one perturbations of self-adjoint operators, where a pure-point spectrum can only stay pure point or become singular continuous. The reason for this contrast comes from the fact that the difference of the self-adjoint operators whose U and V_λ are the Cayley transforms, respectively, is not of rank one in case (ii) of the theorem.

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